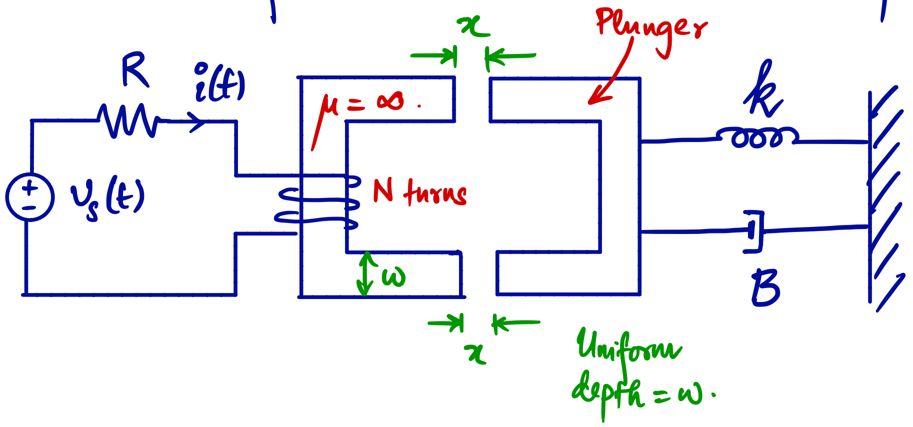
Dynamical system description of electromechanical systems

Next on our agenda is to describe a dynamical system as ordinary differential equations. Then, we will study dynamical systems described by ordinary differential eq^{n}s.

Let's start from an electromechanical system:



Consider the electromechanical system shown above. The plunger is attached to a spring (const. = k), and a dashpot (const. = B), where the spring is uncompressed when $x=0$. Also, a dashpot provides a drag force $B\frac{dx}{dt}$. Finally, ignore fringing.

Let's describe what happens to the system.

- v_s forces current through the conductor.
- Current produces magnetic field, and establishes a flux $\phi = \frac{Ni}{2R_x}$ through the core, plunger and the air gap, where $R_x = \frac{x}{\mu_0 w^2}$.
- Magnetic field magnetizes the plunger.
- Plunger experiences force due to electrical origin.
- The plunger also experiences forces from the spring and the dashpot that guide its motion.

Let's write the electrical & mechanical equations for the system.

Electrical eq^n: $v_s - iR = \dfrac{d\lambda}{dt}$,

where $\lambda = N\dfrac{d\phi}{dt}$, $\phi = \dfrac{Ni}{2R_x}$, and

$R_x = \dfrac{x}{\mu_0 w^2}$.

$$\therefore\ v_s - iR = N \cdot \frac{d}{dt}\left[\frac{\mu_0 w^2 N i}{2x}\right]$$

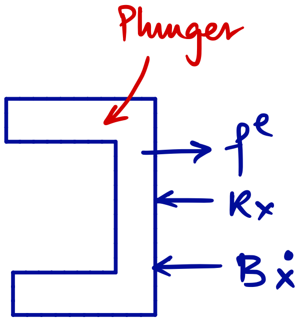
$$= \frac{\mu_0 w^2 N^2}{2} \cdot \frac{d}{dt}\left(\frac{i}{x}\right)$$

$$= \frac{\mu_0 w^2 N^2}{2} \cdot \left[\frac{1}{x}\dot{i} - \frac{i}{x^2} \cdot \dot{x}\right].$$

Here, $\dot{y}$ denotes the time-derivative of any arbitrary variable y.

Mechanical eqⁿ:

Applying Newton's law on the plunger



Free-body diagram of plunger.

$$M\ddot{x} = f^e - Rx - B\dot{x}.$$

Notice ① $\ddot{x}$ = accelaration of plunger. The two dots on top indicate double derivative.

② f^e is always in the direction of positive x.

Calculating f^e:

$$\lambda(i,x) = N\phi(i,x) = \frac{N.\, Ni}{2Rx} = \frac{\mu_0 \omega^2 N^2 i}{2x}.$$

$$\therefore W_m'(i,x) = \int_0^{i} \lambda(\tilde{i},x)\, d\tilde{i}$$

$$= \int \frac{\mu_0 \omega^2 N^2}{2} \cdot \frac{\tilde{i}}{x}\, d\tilde{i} = \frac{\mu_0 \omega^2 N^2 i^2}{4x}.$$

$$f^e = \frac{\partial W_m'}{\partial x} = \frac{\partial}{\partial x}\left[\frac{\mu_0 \omega^2 N^2 i^2}{4x}\right] = -\frac{\mu_0 \omega^2 N^2 i^2}{4x^2}.$$

$\therefore$ mechanical equation of the system is given by

$$M\ddot{x} = -\frac{\mu_0 \omega^2 N^2 i^2}{4x^2} - kx - B\dot{x}.$$

Electromechanical system description :

Electrical equation :

$$v_s - iR = \frac{\mu_0 \omega^2 N^2}{2} \cdot \left[\frac{1}{x}\dot{i} - \frac{i}{x^2}\cdot\dot{x}\right].$$

Mechanical equation :

$$M\ddot{x} = -\frac{\mu_0 \omega^2 N^2 i^2}{4x^2} - kx - B\dot{x}.$$

Let's write these equations in the form $\dot{z} = F(z, u)$, where

z = "states" of the system

u = "inputs" to the system.

States : variables that describe the system completely!

Inputs : External influences that determines how the system evolves.

$\dot{z} = F(z, u)$ ... here, F describes how the states (z) evolve over time.

- Variables that evolve over time in our example : i, x.

For all such variables, identify the highest order of derivative involved in the description.

Electrical equation :

$$v_s - iR = \frac{\mu_0 \omega^2 N^2}{2} \cdot \left[\frac{1}{x} \dot{i} - \frac{i}{x^2} \cdot \dot{x} \right].$$

Mechanical equation :

$$M\ddot{x} = -\frac{\mu_0 \omega^2 N^2 i^2}{4x^2} - kx - B\dot{x}.$$

We have i, $\dot{i}$, x, $\dot{x}$, $\ddot{x}$.

Take the variables **and** their derivatives up to the (highest − 1)-th order and put them in z.

For our example, $Z = \begin{pmatrix} i \\ x \\ \dot{x} \end{pmatrix}$.

- What is u? The external influence in our example is v_s, the voltage source.

Now, let's write the electromechanical eq^n's as $\dot{Z} = F(Z, u)$.

$$\dot{Z} = \begin{pmatrix} \dot{i} \\ \dot{x} \\ \ddot{x} \end{pmatrix}$$ Express each of these as functions of $\begin{pmatrix} i \\ x \\ \dot{x} \end{pmatrix}$ and v_s.

From electrical eq^n, we have

$$\dot{i} = \frac{1}{x}\left[\frac{x^2}{\alpha}(v_s - iR) + i\dot{x}\right]$$

Define $\alpha := \frac{\mu_0 N^2 \omega^2}{2}$, a constant

Notice that $\dot{x}$ is already a part of Z. $\therefore \dot{x} = \dot{x}$.

From mechanical eq^n, we have

$$\ddot{x} = \frac{1}{M}\left[-\frac{\alpha}{2}\frac{i^2}{x^2} - Kx - B\dot{x}\right]$$

$$\dot{i} = \frac{1}{x}\left[\frac{x^2}{\alpha}\left(v_s - iR\right) + i\dot{x}\right]$$

$$\dot{x} = \dot{x}.$$

$$\ddot{x} = \frac{1}{M}\left[-\frac{\alpha}{2}\frac{i^2}{x^2} - kx - B\dot{x}\right]$$

The left-hand side of this eq^n is $\dot{z}$.

The right-hand side of this equation is $F(z, u)$, where $u = v_s$.

General dynamical systems:

$\dot{x} = F_t(x, u)$, $\quad x(t) \in \mathbb{R}^n$ describes the states.

$u(t) \in \mathbb{R}^m$ describes the inputs,

can vary with time!

$F_t : \mathbb{R}^{m \times n} \to \mathbb{R}^n$ describes the dynamics.

<u>Definition</u>: Dimension of the system $= n$, the number of states.

<u>Definition</u>: A dynamical system is said to be linear, if $F : \mathbb{R}^{m \times n} \to \mathbb{R}^n$ is a linear function of x and u.

A linear dynamical system is described by

$$\dot{x}(t) = A(t)\, x(t) + B(t)\, u(t),$$

where $A(t) \in \mathbb{R}^{n \times n}$ and $B(t) \in \mathbb{R}^{n \times m}$ are matrices that may vary with time.

<u>Definition</u>: A system is said to be linear time invariant, if the dynamics does not vary with time, and is described by

$$\dot{x}(t) = A\,x(t) + B\,u(t),$$

where $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$ are matrices that remain constant.

For a dynamical system, we can only study two questions:

- Analysis: How does the system behave?
- Design: How do you design the input for the system towards a certain objective?